NEW IDENTITIES FOR 7-CORES WITH PRESCRIBED BG-RANK

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Abstract. Let π be a partition. BG-rank(π) is defined as an alternating sum of parities of parts of π [1]. In [2], Berkovich and Garvan found theta series representations for the t-core generating functions $\sum_{n\geq 0} a_{t,j}(n)q^n$, where $a_{t,j}(n)$ denotes the number of t-cores of n with BG-rank $= j$. In addition, they found positive *eta*-quotient representations for odd t-core generating functions with extreme values of BG-rank. In this paper we discuss representations of this type for all 7-cores with prescribed BG-rank. We make an essential use of the Ramanujan modular equations of degree 7 [3] to prove a variety of new formulas for the 7-core generating function

$$\prod_{j\geq 1} \frac{(1-q^{7j})^7}{(1-q^j)}.$$

These formulas enable us to establish a number of striking inequalities for $a_{7,j}(n)$ with $j = -1, 0, 1, 2$ and $a_7(n)$, such as

$$a_7(2n+2) \geq 2a_7(n), \quad a_7(4n+6) \geq 10a_7(n).$$

Here $a_7(n)$ denotes a number of unrestricted 7-cores of n. Our techniques are elementary and require creative imagination only.

'Behind every inequality there lies an identity.' – Basil Gordon

Dedicated to our nephews Sam and Yuşa

1. Introduction

A partition $\pi = (\lambda_1, \lambda_2 ..., \lambda_r)$ of n is a nonincreasing sequence of positive integers that sum to n. The BG-rank of π is defined as

$$\text{BG-rank}(\pi) := \sum_{j=1}^{r} (-1)^{j+1} par(\lambda_j), \tag{1.1}$$

where

$$par(\lambda_j) := \begin{cases} 1 & \text{if} \quad \lambda_j \equiv 1 \pmod 2 \\ 0 & \text{if} \quad \lambda_j \equiv 0 \pmod 2 \end{cases}$$

If t is a positive integer, then a partition is a t-core if it has no rim hooks of length t [8]. Let $\pi_{t-\text{core}}$ denote a t-core partition. It is shown in [2, eq.(1.9)] that if t is odd, then

$$-\left\lfloor \frac{t-1}{4} \right\rfloor \leq \text{BG-rank}(\pi_{t-\text{core}}) \leq \left\lfloor \frac{t+1}{4} \right\rfloor. \tag{1.2}$$

Let $a_t(n)$ be the number of t-core partitions of n. It is well known that [9], [5]

$$\sum_{n\geq 0} a_t(n)q^n = \sum_{\vec{n}\in\mathbb{Z}^t,\ \vec{n}.\vec{1_t}=0} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b_t}.\vec{n}} = \frac{(q^t;q^t)^t_\infty}{(q;q)_\infty} = \frac{E^t(q^t)}{E(q)}, \tag{1.3}$$

where

$$\vec{b_t} := (0,1,2,...,t-1), \quad \vec{1_t} := (1,1,...,1), \tag{1.4}$$

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$$(a;q)_n = (a)_n := (1-a)(1-aq)\dots(1-aq^{n-1}),$$
$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1,$$
$$E(q) := (q;q)_\infty.$$

The product $\prod_{i>0} E^{\delta_i}(q^i)$ with $\delta_i \in \mathbb{Z}$ will be referred to as an eta-quotient.

Next, we recall Ramanujan's definition for a general theta function. Let

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab|<1. \tag{1.5}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.6}$$

Two important special cases of (1.5) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{E^5(q^2)}{E^2(q^4)E^2(q)}, \tag{1.7}$$

and

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty = \frac{E^2(q^2)}{E(q)}. \tag{1.8}$$

The product representations in (1.7)–(1.8) are special cases of (1.6). Also, after Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty. \tag{1.9}$$

Let $a_{t,j}(n)$ be the number of t-core partitions of n with BG-rank=j and define their generating function by

$$C_{t,j}(q) := \sum_{n\ge 0} a_{t,j}(n)q^n. \tag{1.10}$$

In this paper, we find representations for $C_{7,0}(q)$ and $C_{7,1}(q)$ in terms of sums of positive eta-quotients. Such representations for $C_{7,2}(q)$ and $C_{7,-1}(q)$ are known (see (1.31)–(1.32) below). Here and throughout the manuscript we say that a q-series is positive if its power series coefficients are nonnegative. We define $P[q]$ to be the set of all such series. Obviously, $\varphi(q)$, $\psi(q)$ and $E^7(q^7)/E(q) \in P[q]$. In fact, Granville and Ono showed that [6] if $t \ge 4$, then $a_t(n) > 0$ for all $n \ge 0$. Our proofs naturally lead us to inequalities that relate the coefficients of $C_{7,j}(q)$, $j = 0, 1, -1, 2$, and to equalities and inequalities for the number of 7-cores. The main results of this paper are organized into two theorems whose proofs are given in sections 4 and 5.

Theorem 1.1. *For all $n \ge 0$, we have*

$$a_7(2n+2) \ge 2a_7(n), \tag{1.11}$$
$$a_7(4n+6) \ge 10a_7(n), \tag{1.12}$$
$$a_{7,0}(n) \ge 9a_{7,2}(n), \tag{1.13}$$
$$a_{7,1}(n) \ge 2a_{7,-1}(n), \tag{1.14}$$
$$a_7(28n+4r) = 5a_7(14n+2r-1),\ r=1,2,6, \tag{1.15}$$
$$a_7(28n+4r+2)+4a_7(7n+r-1) = 5a_7(14n+2r),\ r=2,4,5. \tag{1.16}$$

By equation (1.35) below, we see that (1.12) and (1.13) are equivalent.

Theorem 1.2.

$$C_{7,1}(q) = q\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\big\{\sigma(q^4) + q^2\psi(q^2)\psi(q^{14})\big\}. \tag{1.17}$$

$$\begin{aligned} C_{7,0}(q) = \omega(q^2)\Big\{&\psi^2(q^4)\varphi^2(q^{14}) + q^6\psi^2(q^{28})\varphi^2(q^2) + q^2\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\Big\} \\ &+ q^2\psi(q^4)\psi^2(q^{14})\varphi^3(q^{14}) + 2q^4\psi^3(q^2)\psi^3(q^{14}) + 4q^{12}\psi^2(q^{14})\psi^3(q^{28})\varphi(q^2), \end{aligned} \tag{1.18}$$

where

$$\omega(q) := \psi(q^4)\varphi(q^{14}) + q^3\psi(q^{28})\varphi(q^2) \ \textit{and}\ \sigma(q) := \varphi(q)\varphi(q^7) + 4q^2\psi(q^2)\psi(q^{14}). \tag{1.19}$$

Observe, by (1.6), that

$$\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)} = f(q^2, q^{12})f(q^4, q^{10})f(q^6, q^8)\psi(q^{14}). \tag{1.20}$$

Therefore, each term in (1.17) and (1.18) is a product of six theta functions which are in $P[q]$. It is instructive to compare these representations with those given in (1.27)–(1.28) where for example $C_{7,1}$ is expressed as sum of 21 multi-theta functions.

Our proofs employ the theory of modular equations. The starting point in our proofs is one of Ramanujan's modular equations of degree seven from which we obtained the identity

$$\frac{E^7(q^7)}{E(q)} = f(q, q^{13})f(q^3, q^{11})f(q^5, q^9)\varphi(q^7)\sigma(q^2) + 8q^6\frac{E^7(q^{28})}{E(q^4)}. \tag{1.21}$$

Using several results from Ramanujan's notebooks we obtained the following new analog of (1.21),

$$\frac{E^7(q^7)}{E(q)} = f(q, q^6)f(q^2, q^5)f(q^3, q^4)\psi(q^7)\omega(q) + q^2\frac{E^7(q^{14})}{E(q^2)}. \tag{1.22}$$

The identity (1.22) provided a natural compliment to (1.21) and was essential to our proofs. For proofs of (1.21) and (1.22) see (4.4) and (3.23). From (1.21) and (1.22), we will deduce the following interesting manifestly positive eta-quotient representation for the generating function of 7-cores,

$$\begin{aligned} \frac{E^7(q^7)}{E(q)} =&\sigma(q^4)f(q, q^{13})f(q^3, q^{11})f(q^5, q^9)\varphi(q^7) \\ &+ 2q^3\frac{E^3(q^{28})E^2(q^{14})E^3(q^4)}{E^2(q^2)} + 6q^6\frac{E^7(q^{28})}{E(q^4)} + 2q^2\frac{E(q^{14})^7}{E(q^2)}. \end{aligned} \tag{1.23}$$

Observe, by (1.6), that

$$\frac{E^3(q^{28})E^2(q^{14})E^3(q^4)}{E^2(q^2)} = f(q^2, q^{12})f(q^6, q^8)f(q^4, q^{10})\psi(q^2)\psi^2(q^{14}), \tag{1.24}$$

$$f(q, q^{13})f(q^3, q^{11})f(q^5, q^9)\varphi(q^7) = \frac{\psi(q)\psi(q^7)E^4(q^{14})}{E(q^4)E(q^{28})}. \tag{1.25}$$

The proof of (1.23) is given at the end of section 5.

In [2], it is shown that the generating functions $C_{t,j}(q)$, t odd, can be written as sums of multi-theta functions. We record them here for the case $t = 7$. Let

$$\begin{aligned} B &= (0, 1, 0, 1, 0, 1, 0), \\ \widetilde{B} &= (1, 0, 1, 0, 1, 0, 1). \end{aligned}$$

and for $0 \le i \le 6$ let $\overrightarrow{e}_i$ be the standard unit vector in $\mathbb{Z}^7$. Then

$$C_{7,-1}(q)=\sum_{i=0}^{6}\sum_{\substack{\vec{n}\in\mathbb{Z}^7,\ \vec{n}.\vec{1}_7=0\\ \vec{n}\equiv B+\vec{e}_i\,(\mathrm{mod}\,2\mathbb{Z}^7)}} q^{\frac{7}{2}\|\vec{n}\|^2+\vec{b}_7.\vec{n}}, \tag{1.26}$$

$$C_{7,0}(q)=\sum_{0\le i_0<i_1<i_2\le 6}\ \sum_{\substack{\vec{n}\in\mathbb{Z}^7,\ \vec{n}.\vec{1}_7=0\\ \vec{n}\equiv B+\vec{e}_{i_0}+\vec{e}_{i_1}+\vec{e}_{i_2}\,(\mathrm{mod}\,2\mathbb{Z}^7)}} q^{\frac{7}{2}\|\vec{n}\|^2+\vec{b}_7.\vec{n}}, \tag{1.27}$$

$$C_{7,1}(q)=\sum_{0\le i_0<i_1\le 6}\ \sum_{\substack{\vec{n}\in\mathbb{Z}^7,\ \vec{n}.\vec{1}_7=0\\ \vec{n}\equiv \widetilde{B}+\vec{e}_{i_0}+\vec{e}_{i_1}\,(\mathrm{mod}\,2\mathbb{Z}^7)}} q^{\frac{7}{2}\|\vec{n}\|^2+\vec{b}_7.\vec{n}}, \tag{1.28}$$

$$C_{7,2}(q)=\sum_{\substack{\vec{n}\in\mathbb{Z}^7,\ \vec{n}.\vec{1}_7=0\\ \vec{n}\equiv \widetilde{B}\,(\mathrm{mod}\,2\mathbb{Z}^7)}} q^{\frac{7}{2}\|\vec{n}\|^2+\vec{b}_7.\vec{n}}. \tag{1.29}$$

Eta-quotient representations for

$$C_{t,(-1)^{\frac{t-1}{2}}\lfloor\frac{t}{4}\rfloor}(q) \quad\text{and}\quad C_{t,(-1)^{\frac{t+1}{2}}\lfloor\frac{t+1}{4}\rfloor}(q) \tag{1.30}$$

are obtained in [2, eq.(1.10)–(1.11)]. For $t=7$, they are as follows

$$C_{7,-1}(q)=q^3\frac{E^3(q^{28})E^2(q^{14})E^3(q^4)}{E^2(q^2)}, \tag{1.31}$$

$$C_{7,2}(q)=q^6\frac{E^7(q^{28})}{E(q^4)}. \tag{1.32}$$

As we shall see next, it is easy to find eta-quotient representations for $C_{7,0}(q)$ and $C_{7,1}(q)$ but these representations are not manifestly positive. Observe that if π is a partition of n, then, by definition (1.1),

$$\text{BG-rank}(\pi)\equiv n\,(\mathrm{mod}\,2). \tag{1.33}$$

Therefore, $C_{t,j}(q)$ is either an odd or an even function of q with parity determined by the parity of j. In particular, $C_{7,0}(q)$ and $C_{7,2}(q)$ are even functions of q and $C_{7,1}(q)$ and $C_{7,-1}(q)$ are odd functions of q. Moreover,

$$\sum_{n\ge 0}a_t(n)q^n=\frac{E^7(q^7)}{E(q)}=C_{7,-1}(q)+C_{7,0}(q)+C_{7,1}(q)+C_{7,2}(q). \tag{1.34}$$

Therefore, by (1.32),

$$\begin{aligned} C_{7,0}(q)&=\text{even part of }\left\{\frac{E^7(q^7)}{E(q)}\right\}-C_{7,2}(q)\\ &=\frac{1}{2}\left\{\frac{E^7(q^7)}{E(q)}+\frac{E^7(-q^7)}{E(-q)}\right\}-q^6\frac{E^7(q^{28})}{E(q^4)} \end{aligned} \tag{1.35}$$

and by (1.31),

$$\begin{aligned} C_{7,1}(q)&=\text{odd part of }\left\{\frac{E^7(q^7)}{E(q)}\right\}-C_{7,-1}(q)\\ &=\frac{1}{2}\left\{\frac{E^7(q^7)}{E(q)}-\frac{E^7(-q^7)}{E(-q)}\right\}-q^3\frac{E^3(q^{28})E^2(q^{14})E^3(q^4)}{E^2(q^2)}. \end{aligned} \tag{1.36}$$

The rest of this paper is organized as follows. In the next section, we give a brief introduction to modular equations. Then, we prove three lemmas. In Lemma 3.1, we give several identities for $\sigma(q)$

and $\omega(q)$, which were defined in (1.19). The identity (1.22) in its equivalent form is proved in Lemma 3.2 (see (3.14), (1.20) and (3.23)). These three lemmas are then used in sections 4 and 5 where we prove Theorems 1.1 and 1.2.

2. Modular Equations

In this section, we give background information on modular equations. For $0 < k < 1$, the complete elliptic integral of the first kind $K(k)$, associated with the modulus k, is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The number $k' := \sqrt{1-k^2}$ is called the *complementary modulus*. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2.1}$$

for some positive rational integer n. A relation between k and ℓ induced by (2.1) is called a *modular equation of degree n*. There are several definitions of a modular equation in the literature. For example, see the books by R. A. Rankin [10, p. 76] and B. Schoeneberg [11, pp. 141–142]. Following Ramanujan, set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = \ell^2.$$

We often say that β has degree n over α. If

$$q = \exp(-\pi K'/K), \tag{2.2}$$

two of the most fundamental relations in the theory of elliptic functions are given by the formulas [3, pp. 101–102],

$$\varphi^2(q) = \frac{2}{\pi}K(k) \text{ and } \alpha = k^2 = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{2.3}$$

The equation (2.3) and elementary theta function identities make it possible to write each modular equation as a theta function identity. Ramanujan derived an extensive "catalogue" of formulas [3, pp. 122–124] giving the "evaluations" of $E(q)$, $\varphi(q)$, $\psi(q)$, and $\chi(q)$ at various powers of the arguments in terms of

$$z := z_1 := \frac{2}{\pi}K(k), \quad \alpha, \quad \text{and} \quad q.$$

The evaluations that will be needed in this paper are as follows,

$$\varphi(q) = \sqrt{z}, \tag{2.4}$$

$$\varphi(-q) = \sqrt{z}(1-\alpha)^{1/4}, \tag{2.5}$$

$$\varphi(-q^2) = \sqrt{z}(1-\alpha)^{1/8}, \tag{2.6}$$

$$\psi(q) = q^{-1/8}\sqrt{\tfrac{1}{2}z}\alpha^{1/8}, \tag{2.7}$$

$$\psi(-q) = q^{-1/8}\sqrt{\tfrac{1}{2}z}\{\alpha(1-\alpha)\}^{1/8}, \tag{2.8}$$

$$\psi(q^2) = 2^{-1}q^{-1/4}\sqrt{z}\alpha^{1/4}, \tag{2.9}$$

$$E(-q) = 2^{-1/6}q^{-1/24}\sqrt{z}\{\alpha(1-\alpha)\}^{1/24}, \tag{2.10}$$

$$E(q^2) = 2^{-1/3}q^{-1/12}\sqrt{z}\{\alpha(1-\alpha)\}^{1/12}, \tag{2.11}$$

$$\chi(-q^2) = 2^{1/3}q^{1/12}\alpha^{-1/12}(1-\alpha)^{1/24}. \tag{2.12}$$

We should remark that in the notation of [3], $E(q) = f(-q)$. If q is replaced by q^n, then the evaluations are given in terms of

$$z_n := \frac{2}{\pi}K(l), \quad \beta, \quad \text{and} \quad q^n,$$

where β has degree n over α.

Lastly, the multiplier m of degree n is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)} = \frac{z}{z_n}. \tag{2.13}$$

The proofs of the following modular equations of degree seven can be found in [3, p. 314, Entry 19(i),(iii)],

$$(\alpha\beta)^{1/8} + \big\{(1-\alpha)(1-\beta)\big\}^{1/8} = 1, \tag{2.14}$$

$$\Big(\tfrac{1}{2}\big(1 + (\alpha\beta)^{1/8} + \big\{(1-\alpha)(1-\beta)\big\}^{1/8}\big)\Big)^{1/2} = 1 - \big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{1/8}, \tag{2.15}$$

$$\Big(\frac{(1-\beta)^7}{(1-\alpha)}\Big)^{1/8} - \Big(\frac{\beta^7}{\alpha}\Big)^{1/8} = m\Big(\tfrac{1}{2}\Big(1 + (\alpha\beta)^{1/8} + \Big\{(1-\alpha)(1-\beta)\Big\}^{1/8}\Big)\Big)^{1/2}, \tag{2.16}$$

$$m = \frac{1 - 4\Big(\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\Big)^{1/24}}{\big\{(1-\alpha)(1-\beta)\big\}^{\frac{1}{8}} - (\alpha\beta)^{\frac{1}{8}}}, \quad \frac{7}{m} = -\frac{1 - 4\Big(\dfrac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\Big)^{1/24}}{\big\{(1-\alpha)(1-\beta)\big\}^{\frac{1}{8}} - (\alpha\beta)^{\frac{1}{8}}}, \tag{2.17}$$

$$\Big(\frac{(1-\beta)^7}{(1-\alpha)}\Big)^{1/8} + \Big(\frac{\beta^7}{\alpha}\Big)^{1/8} + 2\Big(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\Big)^{1/24} = \frac{3+m^2}{4}. \tag{2.18}$$

3. Three Lemmas

Lemma 3.1. *If $\sigma(q)$ and $\omega(q)$ are defined by* (1.19)*, then*

$$\sigma(q^2) = \varphi(q)\varphi(q^7) - 2q\psi(-q)\psi(-q^7), \tag{3.1}$$

$$\sigma(q) = \sigma(q^2) + 2q\psi(q)\psi(q^7), \tag{3.2}$$

$$\omega^2(q) = \psi(q)\psi(q^7)\big(\sigma(q^2) - q\psi(q)\psi(q^7)\big), \tag{3.3}$$

$$\sigma^2(q^2) = 4q\omega^2(q) + \varphi^2(-q)\varphi^2(-q^7). \tag{3.4}$$

Proof. We start with two identities from [3, pp. 304, 315, eq. (19.1)],

$$\varphi(-q^2)\varphi(-q^{14}) = \varphi(-q)\varphi(-q^7) + 2q\psi(-q)\psi(-q^7), \tag{3.5}$$

$$\psi(q)\psi(q^7) = \psi(q^8)\varphi(q^{28}) + q^6\psi(q^{56})\varphi(q^4) + q\psi(q^2)\psi(q^{14}). \tag{3.6}$$

We will frequently use (3.6) in the form

$$\psi(q)\psi(q^7) = \omega(q^2) + q\psi(q^2)\psi(q^{14}). \tag{3.7}$$

Using the well-known identity, [3, p. 40, Entry 25 9(i),(ii)]

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8),$$

it is easily verified that

$$\begin{aligned}\varphi(q)\varphi(q^7) &= \varphi(q^4)\varphi(q^{28}) + 4q^8\psi(q^8)\psi(q^{56}) \\ &\quad + 2q\{\psi(q^8)\varphi(q^{28}) + q^6\psi(q^{56})\varphi(q^4)\} \\ &= \sigma(q^4) + 2q\omega(q^2).\end{aligned} \tag{3.8}$$

Using (3.7) and (3.8) in (3.5), we find that

$$\varphi(-q^2)\varphi(-q^{14}) = \sigma(q^4) - 2q\omega(q^2) + 2q\omega(q^2) - 2q^2\psi(q^2)\psi(q^4). \tag{3.9}$$

Replacing $-q^2$ by q, we conclude that

$$\sigma(q^2)=\varphi(q)\varphi(q^7)-2q\psi(-q)\psi(-q^7), \tag{3.10}$$

which is (3.1). Similarly, using (3.7) and (3.8) in (3.10), we arrive at

$$\begin{aligned}\sigma(q^2)&=\varphi(q)\varphi(q^7)-2q\psi(-q)\psi(-q^7)\\&=\sigma(q^4)+2q\omega(q^2)-2q\omega(q^2)+2q^2\psi(q^2)\psi(q^{14})\\&=\sigma(q^4)+2q^2\psi(q^2)\psi(q^{14}),\end{aligned} \tag{3.11}$$

which is (3.2) with q replaced by q^2. Lastly, by (3.7), (3.8), and by the trivial identity $\psi^2(q)=\psi(q^2)\varphi(q)$, we find that

$$\begin{aligned}4\omega^2(q^2)=&\big(\psi(q)\psi(q^7)+\psi(-q)\psi(-q^7)\big)^2\\=&\,\psi^2(q)\psi^2(q^7)+\psi^2(-q)\psi^2(-q^7)+2\psi(q)\psi(q^7)\psi(-q)\psi(-q^7)\\=&\,\psi(q^2)\psi(q^{14})\big(\varphi(q)\varphi(q^7)+\varphi(-q)\varphi(-q^7)\big)\\&+2\big(\omega(q^2)+q\psi(q^2)\psi(q^{14})\big)\big(\omega(q^2)-q\psi(q^2)\psi(q^{14})\big)\\=&\,2\psi(q^2)\psi(q^{14})\sigma(q^4)+2\omega^2(q^2)-2q^2\psi^2(q^2)\psi^2(q^{14}),\end{aligned} \tag{3.12}$$

from which (3.3) immediately follows.

The identity (3.4), which is not employed in this manuscript, was first proven in [4]. Here we provide a short new proof. By (3.1) with q replaced by $-q$, we find that

$$\begin{aligned}&\sigma^2(q^2)-\varphi^2(-q)\varphi^2(-q^7)\\&=\Big(\sigma(q^2)-\varphi(-q)\varphi(-q^7)\Big)\Big(\sigma(q^2)+\varphi(-q)\varphi(-q^7)\Big)\\&=2q\psi(q)\psi(q^7)\Big(\sigma(q^2)+\varphi(-q)\varphi(-q^7)\Big).\end{aligned} \tag{3.13}$$

Now, by (3.13), (3.3), and (3.1) with q replaced by $-q$, we deduce that

$$\begin{aligned}&\sigma^2(q^2)-\varphi^2(-q)\varphi^2(-q^7)-4q\omega^2(q)\\&=2q\psi(q)\psi(q^7)\Big(\sigma(q^2)+\varphi(-q)\varphi(-q^7)-2\sigma(q^2)+2q\psi(q)\psi(q^7)\Big)\\&=0,\end{aligned}$$

which is (3.4).

□

Lemma 3.2. *With $\omega(q)$ defined by* (1.19),

$$f(q,q^6)f(q^2,q^5)f(q^3,q^4)=q^2\psi^3(q^7)+\psi(q)\omega(q). \tag{3.14}$$

Proof. By (1.6), we find that

$$f(q,q^6)f(q^2,q^5)f(q^3,q^4)=\frac{(-q;q)_\infty}{(-q^7;q^7)_\infty}E^3(q^7)=\frac{\chi(-q^7)}{\chi(-q)}E^3(q^7). \tag{3.15}$$

In (3.14), if we replace q by q^2, and use (3.7), and (3.15) with q replaced by q^2, we are led to prove

$$\frac{\chi(-q^{14})}{\chi(-q^2)}E^3(q^{14})=q^4\psi^3(q^{14})+\psi(q^2)\big\{\psi(q)\psi(q^7)-q\psi(q^2)\psi(q^{14})\big\}. \tag{3.16}$$

Transforming (3.16) by means of the evaluations given by (2.12), (2.11), (2.9) and (2.7), we find that

$$\begin{aligned}&\frac{1}{2}q^{-5/4}\sqrt{z_7^3}\frac{\alpha^{1/12}\beta^{1/6}(1-\beta)^{7/4}}{(1-\alpha)^{1/24}}\\&=\frac{1}{8}q^{-5/4}\sqrt{z_7^3}\beta^{3/4}+\frac{1}{2}q^{-1/4}\sqrt{z_1}\alpha^{1/4}\big\{\frac{1}{2}q^{-1}\sqrt{z_1z_7}(\alpha\beta)^{1/8}-\frac{1}{4}q^{-1}\sqrt{z_1z_7}(\alpha\beta)^{1/4}\big\}.\end{aligned}$$

Simplifying and using (2.13), we arrive at

$$4\Big(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\Big)^{1/24}=\Big(\frac{\beta^7}{\alpha}\Big)^{1/8}+m(\alpha\beta)^{1/8}\big\{2(\alpha\beta)^{1/8}-(\alpha\beta)^{1/4}\big\}. \tag{3.17}$$

Set $t:=(\alpha\beta)^{1/8}$. Then, by (2.14), we have

$$\big\{(1-\alpha)(1-\beta)\big\}^{1/8}=1-t. \tag{3.18}$$

The equation (3.17) now takes the form

$$4\Big\{\frac{\beta(1-\beta)}{t(1-t)}\Big\}^{1/3}=\frac{\beta}{t}+mt(2t-t^2). \tag{3.19}$$

It is shown in [3, pp. 319–320, eqs. (19.19), (19.21)] that

$$m=\frac{t-\beta}{t(1-t)(1-t+t^2)}, \tag{3.20}$$

and

$$(1-2t)m=1-4\Big(\frac{\beta(1-\beta)}{t(1-t)}\Big)^{1/3}. \tag{3.21}$$

Using (3.21) in the left-hand side of (3.19) and solving for m, we obtain (3.20). Hence, the proof of (3.14) is complete. □

We now make several observations which will be used later. By (3.14) and by (1.20) with q^2 replaced by q, we find that

$$\frac{E(q^{14})E^3(q^7)E(q^2)}{E(q)}=q^2\psi^4(q^7)+\psi(q)\psi(q^7)\omega(q). \tag{3.22}$$

Multiplying both sides of (3.22) by $\dfrac{E^4(q^7)}{E(q^2)E(q^{14})}$, we conclude that

$$\frac{E^7(q^7)}{E(q)}=q^2\frac{E^7(q^{14})}{E(q^2)}+\frac{E(q^{14})E^3(q^7)E(q^2)}{E(q)}\omega(q), \tag{3.23}$$

which, by (1.20), is equivalent to (1.22).

We should remark that if β has degree seven over α, then α, β and the *multiplier* m can be written as rational functions of the parameter $t=(\alpha\beta)^{1/8}$ [3, pp. 316–319]. This parametrization is a very efficient tool in verifying modular equations of degree seven.

Lemma 3.3.

$$\frac{1}{2}\Big\{\frac{E^7(q^7)}{E(q)}+\frac{E^7(-q^7)}{E(-q)}\Big\}=5q^2\frac{E^7(q^{14})}{E(q^2)}-4q^6\frac{E^7(q^{28})}{E(q^4)}+E^3(q^2)E^3(q^{14}). \tag{3.24}$$

Proof. From (2.17), we find that

$$7\Big(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\Big)^{1/24}+m^2\Big(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\Big)^{1/24}=\frac{m^2+7}{4}. \tag{3.25}$$

Upon comparison with (2.18), we conclude that

$$5\Big(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\Big)^{1/24}+m^2\Big(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\Big)^{1/24}=\Big(\frac{(1-\beta)^7}{(1-\alpha)}\Big)^{1/8}+\Big(\frac{\beta^7}{\alpha}\Big)^{1/8}+1. \tag{3.26}$$

Transforming (3.26) by means of the evaluations given by (2.10), (2.6) and (2.7), we find that

$$10q^2\frac{\sqrt{z}}{\sqrt{z_7}^7}\frac{E^7(-q^7)}{E(-q)}+2m^2\frac{\sqrt{z_7}}{\sqrt{z}^7}\frac{E^7(-q)}{E(-q^7)}=\frac{\sqrt{z}}{\sqrt{z_7}^7}\frac{\varphi^7(-q^{14})}{\varphi(-q^2)}+8q^6\frac{\sqrt{z}}{\sqrt{z_7}^7}\frac{\psi^7(q^7)}{\psi(q)}+1. \tag{3.27}$$

Multiplying both sides of (3.27) by $\dfrac{\sqrt{z}}{\sqrt{z_7}^7}\dfrac{(-q,q^2)_\infty}{(-q^7,q^{14})^7_\infty}$ and using (2.13), we obtain (3.24). □

An interesting corollary of (3.24) will be given at the end of the next section. We should add that (3.24) can be rewritten as

$$T_2\big(q^2\frac{E^7(q^7)}{E(q)}\big)= 5q^2\frac{E^7(q^7)}{E(q)}+qE^3(q)E^3(q^7), \tag{3.28}$$

where, the Hecke operator T_2 is defined by

$$T_2\big(\sum a(n)q^n\big)=\sum(a(2n)+4a(n/2))q^n, \tag{3.29}$$

with $a(n/2)=0$ if n is odd.

4. Proof of Theorem 1.1

By (2.15) and (2.16), we have

$$\Big(\frac{(1-\beta)^7}{(1-\alpha)}\Big)^{1/8}-\Big(\frac{\beta^7}{\alpha}\Big)^{1/8}=m(1-\big\{\alpha\beta(1-\alpha)(1-\beta)\big\}^{1/8}). \tag{4.1}$$

Transforming (4.1) by means of the evaluations given by (2.6)–(2.8), we find that

$$\frac{\sqrt{z}}{\sqrt{z_7}^7}\frac{\varphi^7(-q^{14})}{\varphi(-q^2)}-8\frac{\sqrt{z}}{\sqrt{z_7}^7}q^6\frac{\psi^7(q^7)}{\psi(q)}=\frac{z}{z_7}\Big(1-\frac{2}{\sqrt{z}\sqrt{z_7}}q\psi(-q)\psi(-q^7)\Big). \tag{4.2}$$

Simplifying, and using (2.4) and (3.1), we conclude that

$$\frac{\varphi^7(-q^{14})}{\varphi(-q^2)}-8q^6\frac{\psi^7(q^7)}{\psi(q)}=\varphi^4(q^7)\Big\{\varphi(q)\varphi(q^7)-2q\psi(-q)\psi(-q^7)\Big\}=\varphi^4(q^7)\sigma(q^2). \tag{4.3}$$

Multiplying both sides of (4.3) by $\dfrac{(-q,q^2)_\infty}{(-q^7,q^{14})^7_\infty}$, we find that

$$\frac{E^7(q^7)}{E(q)}-8q^6\frac{E^7(q^{28})}{E(q^4)}=\frac{\psi(q)\psi(q^7)E^4(q^{14})}{E(q^4)E(q^{28})}\sigma(q^2), \tag{4.4}$$

which, by (1.25), is equivalent to (1.21).

Next, by (3.7), (3.2), and by (3.23), we see that

$$\begin{aligned}
&\text{even part of}\Big\{\frac{E^7(q^7)}{E(q)}\Big\}\\
&=\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)\sigma(q^2)+8q^6\frac{E^7(q^{28})}{E(q^4)}\\
&=\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)\big(\sigma(q^4)+2q^2\psi(q^2)\psi(q^{14})\big)+8q^6\frac{E^7(q^{28})}{E(q^4)}\\
&=\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)\sigma(q^4)+2q^2\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\omega(q^2)+8q^6\frac{E^7(q^{28})}{E(q^4)}\\
&=\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)\sigma(q^4)+2q^2\Big\{\frac{E^7(q^{14})}{E(q^2)}-q^4\frac{E^7(q^{28})}{E(q^4)}\Big\}+8q^6\frac{E^7(q^{28})}{E(q^4)}\\
&=2q^2\frac{E^7(q^{14})}{E(q^2)}+6q^6\frac{E^7(q^{28})}{E(q^4)}+\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)\sigma(q^4).
\end{aligned} \tag{4.5}$$

Recall that we defined $P[q]$ to be the set of all q-series with non-negative coefficients. Now, by (3.7) and (1.25),

$$\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)=\text{even part of}\Big\{\frac{\psi(q)\psi(q^7)E^4(q^{14})}{E(q^4)E(q^{28})}\Big\}\in P[q] \tag{4.6}$$

Therefore, we conclude

$$\frac{E^7(q^7)}{E(q)} - 2q^2\frac{E^7(q^{14})}{E(q^2)} \in P[q], \tag{4.7}$$

which is clearly equivalent to (1.11). Alternatively, one can directly establish that

$$\frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2) = f(q^4,q^{24})f^3(q^{12},q^{16}) + q^6 f(q^{10},q^{18})f^3(q^2,q^{26}) \in P[q]. \tag{4.8}$$

We will not use (4.8), and so we forgo its proof.

From (4.5), we have

$$\frac{E^7(q^7)}{E(q)} = 2q^2\frac{E^7(q^{14})}{E(q^2)} + 6q^6\frac{E^7(q^{28})}{E(q^4)} + s(q), \tag{4.9}$$

where $s(q) \in P[q]$. Iterating (4.9), we find that

$$\begin{aligned}\frac{E^7(q^7)}{E(q)} &= 2q^2\Big(2q^4\frac{E^7(q^{28})}{E(q^4)} + 6q^{12}\frac{E^7(q^{56})}{E(q^8)} + s(q^2)\Big)+6q^6\frac{E^7(q^{28})}{E(q^4)} + s(q)\\ &= 10q^6\frac{E^7(q^{28})}{E(q^4)} + s_1(q),\end{aligned} \tag{4.10}$$

where $s_1(q) \in P[q]$. This last identity clearly implies (1.12). We already remarked that, by equation (1.35), (1.12) and (1.13) are equivalent.

To prove (1.14) we return to (4.4). We have by (3.7), (1.31), (3.2) and by (3.3)

$$\begin{aligned}&\text{odd part of}\Big\{\frac{E^7(q^7)}{E(q)}\Big\}-3C_{7,-1}(q)\\ &= q\frac{\psi(q^2)\psi(q^{14})E^4(q^{14})}{E(q^4)E(q^{28})}\sigma(q^2) - 3q^3\frac{E^3(q^{28})E^2(q^{14})E^3(q^4)}{E^2(q^2)}\\ &= q\frac{E(q^4)E(q^{28})E^3(q^{14})}{E(q^2)}\Big\{\sigma(q^2) - 3q^2\psi(q^2)\psi(q^{14})\Big\}\\ &= q\frac{E(q^4)E(q^{28})E^3(q^{14})}{E(q^2)}\Big\{\sigma(q^4) - q^2\psi(q^2)\psi(q^{14})\Big\}\\ &= q\omega^2(q^2)\frac{E^4(q^{14})}{E(q^4)E(q^{28})}.\end{aligned} \tag{4.11}$$

By (4.6), we see that

$$\text{odd part of}\Big\{\frac{E^7(q^7)}{E(q)}\Big\}-3C_{7,-1}(q) \in P[q], \tag{4.12}$$

which, by (1.36), is clearly equivalent to (1.14).

Lastly, we prove (1.15) and (1.16). Let $b(n)$ be defined by

$$\sum_{n\geq 0} b(n)q^n = E^3(q)E^3(q^7). \tag{4.13}$$

From (3.24) with q^2 replaced by q, we find that

$$\sum a_7(2n)q^n = 5q\sum a_7(n)q^n - 4q^3\sum a_7(n)q^{2n} + \sum b(n)q^n. \tag{4.14}$$

Equating the even indexed terms in both sides of (4.14), we arrive at

$$a_7(4n) - 5a_7(2n-1) = b(2n). \tag{4.15}$$

Using Jacobi's well-known identity for $E^3(q)$ [7, Thm. 357], namely,

$$E^3(q) = \sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)q^{k(k-1)/2}, \tag{4.16}$$

we easily conclude that $b(n)=0$ if $n\equiv 2,4,5 \pmod 7$. This observation together with (4.14) implies (1.15). The equation (1.16) is proved similarly by equating the odd indexed terms in both sides (4.14).

Corollary 4.1.

$$3a_7(n-1)+b(n)\geq 0 \textit{ for all } n>0. \tag{4.17}$$

Proof. By (4.5), we can write (3.24) in its equivalent form

$$3q\frac{E^7(q^7)}{E(q)}+E^3(q)E^3(q^7)=10q^3\frac{E^7(q^{14})}{E(q^2)}+\sigma(q^2)\omega(q)\frac{E^4(q^7)}{E(q^2)E(q^{14})}. \tag{4.18}$$

By (4.6), we see that the right-hand side of (4.18) is in $P[q]$, from which (4.17) is immediate. □

5. Proof of Theorem 1.2 and (1.23)

By (1.36), (4.4), (3.7), (1.31) and by (3.2), we have that

$$\begin{aligned}
C_{7,1}(q) &= \text{odd part of}\Big\{\frac{E^7(q^7)}{E(q)}\Big\}-C_{7,-1}(q)\\
&= q\frac{\psi(q^2)\psi(q^{14})E^4(q^{14})}{E(q^4)E(q^{28})}\sigma(q^2)-q^3\frac{E^3(q^{28})E^2(q^{14})E^3(q^4)}{E^2(q^2)}\\
&= q\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\Big\{\sigma(q^2)-q^2\psi(q^2)\psi(q^{14})\Big\}\\
&= q\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\Big\{\sigma(q^4)+q^2\psi(q^2)\psi(q^{14})\Big\}.
\end{aligned} \tag{5.1}$$

This completes the proof of (1.17).

Next, we prove (1.18). Combining (3.22) and (3.23), we have

$$\frac{E^7(q^7)}{E(q)}=q^2\frac{E^7(q^{14})}{E(q^2)}+q^2\psi^4(q^7)\omega(q)+\psi(q)\psi(q^7)\omega^2(q). \tag{5.2}$$

Using (3.23) with q replaced by q^2 in (5.2), we find that

$$\begin{aligned}
\frac{E^7(q^7)}{E(q)} &= q^2\Big\{q^4\frac{E^7(q^{28})}{E(q^4)}+\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\omega(q^2)\Big\}+q^2\psi^4(q^7)\omega(q)+\psi(q)\psi(q^7)\omega^2(q)\\
&= q^6\frac{E^7(q^{28})}{E(q^4)}+q^2\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\omega(q^2)+q^2\psi^4(q^7)\omega(q)+\psi(q)\psi(q^7)\omega^2(q).
\end{aligned} \tag{5.3}$$

It now remains to find the even part of the last two terms on the right side of (5.3). This is is easily done with the even-odd dissections of $\omega(q)$ and $\psi(q)\psi(q^7)$ given by (1.19) and (3.7) and the formula (see [3, p. 40, Entry 25 (iv)–(vii)])

$$\psi^4(q)=\psi^2(q^2)(\varphi^2(q^2)+4q\psi^2(q^4)) \tag{5.4}$$

with q replaced by q^7.

Lastly, we prove (1.23). Arguing as in (4.11), we find that

$$\begin{aligned}
&\text{odd part of}\Big\{\frac{E^7(q^7)}{E(q)}\Big\}-2C_{7,-1}(q)\\
&= q\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\Big\{\sigma(q^2)-2q^2\psi(q^2)\psi(q^{14})\Big\}\\
&= q\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\sigma(q^4),
\end{aligned} \tag{5.5}$$

where in the last step, we used (3.2). Using (5.5) together with (4.5), and by (3.7) and (1.25), we arrive at

$$\begin{aligned}\frac{E^7(q^7)}{E(q)} &= q\frac{E(q^{28})E^3(q^{14})E(q^4)}{E(q^2)}\sigma(q^4) + 2C_{7,-1}(q) + 2q^2\frac{E^7(q^{14})}{E(q^2)} \\ &\quad + 6q^6\frac{E^7(q^{28})}{E(q^4)} + \frac{E^4(q^{14})}{E(q^4)E(q^{28})}\omega(q^2)\sigma(q^4) \\ &= 2C_{7,-1}(q) + 2q^2\frac{E^7(q^{14})}{E(q^2)} + 6q^6\frac{E^7(q^{28})}{E(q^4)} + \frac{E^4(q^{14})}{E(q^4)E(q^{28})}\sigma(q^4)\Big\{\omega(q^2) + q\psi(q^2)\psi(q^{14})\Big\} \\ &= 2C_{7,-1}(q) + 2q^2\frac{E^7(q^{14})}{E(q^2)} + 6q^6\frac{E^7(q^{28})}{E(q^4)} + \frac{E^4(q^{14})}{E(q^4)E(q^{28})}\sigma(q^4)\psi(q)\psi(q^7) \\ &= 2C_{7,-1}(q) + 2q^2\frac{E^7(q^{14})}{E(q^2)} + 6q^6\frac{E^7(q^{28})}{E(q^4)} + \sigma(q^4)f(q,q^{13})f(q^3,q^{11})f(q^5,q^9)\varphi(q^7), \end{aligned} \tag{5.6}$$

which, by (1.31), is equal to the right hand side of (1.23).

6. Concluding Remarks

The inequalities , (1.11) and (1.12) (or equivalently (1.13)), of Theorem 1.1 are not optimal. Numerical evidence suggest that

$$\begin{aligned} a_7(2n+2) &\geq 3a_7(n) \quad \text{for all} \quad n \geq 1, \\ a_7(4n+6) &\geq 15a_7(n) \quad \text{for all} \quad n \geq 1, \\ a_7(4n+6) &\geq 11a_7(n) \quad \text{for all} \quad n \geq 0. \end{aligned}$$

Our attempts to improve Theorems 1.1 and 1.2 led us to the following interesting conjectures:

$$\psi(q)\big(\psi^2(q) - \psi^2(q^7)\big) \in P[q], \tag{6.1}$$

$$\psi(q)\big(\varphi^2(q) - \varphi^2(q^7)\big) \in P[q], \tag{6.2}$$

$$\varphi(q)\big(\psi^2(q) - \psi^2(q^7)\big) \in P[q], \tag{6.3}$$

and

$$\psi(q)\big(\varphi^2(q) - \psi^2(q^7)\big) \in P[q]. \tag{6.4}$$

Referee pointed out that (1.15) and (1.16) extend easily using our arguments to a few other arithmetic progressions; for example,

$$a_7(196n + 4r) = 5a_7(98n + 2r - 1) \quad \text{for } r = 10, 17, 45.$$

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